数理解析研究所講究録
第 864 巻 1994 年 124-133

Observations* on Conditions Assuring $\mathbf{int}A + B = \mathbf{int}(A + B)$

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Abstract. The aim of this paper is to call attention to some elementary properties of convex sets for the core operator and the interior operator in a linear space and a linear topological space, respectively.

Key Words. Convex analysis, optimization, convex set, interior point, midconvex set.

1. INTRODUCTION AND PRELIMINARIES

The concept of convexity is very important in various fields of mathematics as well as the area of applied mathematics. The origin of interest in convexity arises from areas of application related to fixed point theory and optimization theory. Nowadays, most of basic properties of convex sets are found in much literature, and we can also find many books which are concerned with convex analysis. Nevertheless, to the best of the authors' knowledge, the elementary property as in this paper's title is not taken up in those books except for special cases; e.g., Corollary 6.6.2 in [5] limited to R^n. Hence, the aim of this paper is to prove such property of convexity and call attention to it. For this end we will show some properties of convex sets for the core operator and the interior operator in a linear space and a linear topological space, respectively. Also, we will observe some family on which each operator behaves like a linear mapping. Moreover, we will show that $\mathrm{cor}A + B$ is midconvex if and only if $\mathrm{cor}A + B$ is convex.

Throughout this paper, the term linear space will refer to a linear space over the real field R or over the complex field C. Given a linear space X, and $a, b \in X, a \neq b$, we will use the following notation for line segment subsets (joining a and b) of X; $[a, b] := \{\lambda a + (1-\lambda)b : 0 \leq \lambda \leq 1\}$, $[a, b) := [a, b] \setminus \{b\}$, and $(a, b) := [a, b) \setminus \{a\}$. A subset A of X is said to be convex if for every $a, b \in A$, $a \neq b$, the line segment $[a, b]$ is a subset of A. Also, a subset A of X is said to be midconvex if for every $a, b \in A$, $\frac{1}{2}(a + b) \in A$. Of course, any convex

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*AMS 1991 Subject Classifications. 52A07. The authors are very grateful to Professor J. Nakagami of Chiba University for this opportunity of giving a talk about this work.

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set is also midconvex. We usually define addition and scalar multiplication on the family $P(X)$ of nonempty subsets of X as follows:

$$A + B := \{a + b : a \in A, b \in B\},$$

$$\lambda A := \{\lambda a : a \in A\},$$

where $A, B \in P(X)$ and λ is a scalar. Then, we recall that for a subset C of X, a subset A of X is said to be C-convex if $A + C$ is a convex set. This concept is usually used in vector optimization and multiobjective programming when the set C is a convex cone. Also, we will use the following symbols: $x + A := \{x + a : a \in A\}$, where $x \in X$. The addition and scalar multiplication do not define a linear space structure on $P(X)$, but we can define an operator which behaves like a linear mapping on $P(X)$. For example, it is well-known that

$$\mathrm{co}(\alpha A + \beta B) = \alpha \mathrm{co}(A) + \beta \mathrm{co}(B),$$

where $A, B \in P(X)$ and α, β are scalars, and $\mathrm{co}(A)$ denotes the convex hull of the set $A \in P(X)$; see page 6 in [2]. In this paper, we will show that each of the core operator and the interior operator behaves like a linear mapping on the family $C(X)$ of nonempty convex subsets of X. Of course, the convex hull operator co behaves like an identity on $C(X)$. Recall that $A + B$ and λA are convex for any $A, B \in C(X)$ and scalar λ, that is, $C(X)$ behaves like a linear subspace of $P(X)$ except for some axioms.

Next, we define the core operator, the interior operator, and the closure operator. For a subset A of a linear space X, the core of A (or algebraic interior of A), written $\mathrm{cor}A$, is the set of all points $a \in A$ such that for each $x \in X \setminus \{a\}$ there exists $b \in (a, x)$ for which $[a, b] \subset A$. For a subset A of a linear topological space $(X, \mathcal{T})$, the interior of A (or topological interior of A), written $\mathrm{int}A$, is defined by $\mathrm{int}A := \cup\{U \in \mathcal{T} : U \subset A\}$. It is clearly open and is the largest open set included in A. Also, for a subset A of a linear topological space $(X, \mathcal{T})$, the closure of A (or topological closure of A), written $\mathrm{cl}A$, is defined by $\mathrm{cl}A := \cap\{F \subset X : F \supset A \text{ and } F \text{ is closed }\}$. It is clearly closed and is the smallest closed set including A.

Remark 1.1. We give some elementary properties of convex sets, which we shall need later. First, it is clear that for any $A \in P(X)$ and scalar λ, $\mathrm{cor}(\lambda A) = \lambda \mathrm{cor}A$, $\mathrm{int}(\lambda A) = \lambda \mathrm{int}A$ (whenever $\mathrm{cor}A \neq \emptyset$, $\mathrm{int}A \neq \emptyset$, and $\lambda \neq 0$), $\mathrm{cl}(\lambda A) = \lambda \mathrm{cl}A$, and also $\mathrm{int}(\mathrm{int}A) = \mathrm{int}A$, $\mathrm{cl}(\mathrm{cl}A) = \mathrm{cl}A$. Second, remark that $\mathrm{cor}A$, $\mathrm{int}A$, and $\mathrm{cl}A$ are convex (or empty) whenever A is convex. Finally, let $C_0(X) := \{A \in C(X) : \mathrm{int}A \neq \emptyset\}$ then $\mathrm{int}(\mathrm{cl}A) = \mathrm{int}A$ and $\mathrm{cl}(\mathrm{int}A) = \mathrm{cl}A$ for any $A \in C_0(X)$; see page 59 in [2].

2. ADDITIVITY OF CORE OPERATOR AND INTERIOR OPERATOR

At first, we observe some properties of the core operator and the interior operator without convexity. Those properties are elementary but for the reader's convenience we will give the proofs.

Proposition 2.1. *Let A and B be nonempty sets in a linear space X. If $\mathrm{cor}A \neq \emptyset$ then*

$$\mathrm{cor}A + B \subset \mathrm{cor}(A + B). \tag{2.1}$$

Proof. For any $x \in \mathrm{cor}A + B$, there are $a \in \mathrm{cor}A$ and $b \in B$ such that $x = a + b$. Then for each $z \in X \setminus \{x\}$, there exists $y \in (a, z - b)$ such that $[a, y] \subset A$. Hence, $[x, y + b] = [a, y] + b \subset A + B$ and $y + b \in (x, z)$. Therefore, we have $x \in \mathrm{cor}(A + B)$. ∎

Proposition 2.2. (See page 40 in [1].) *Let A and B be nonempty sets in a linear topological space X. If* $\mathrm{int}A \neq \emptyset$ *then*

$$\mathrm{int}A + B \subset \mathrm{int}(A + B). \tag{2.2}$$

Proof. Since for any non-zero scalar α and vector $x_0 \in X$ the map $x \mapsto x_0 + \alpha x$ is a homeomorphism of X with itself, the set

$$\mathrm{int}A + B = \bigcup_{b \in B} (\mathrm{int}A + b)$$

is open and contained in $A + B$. Hence, $\mathrm{int}A + B \subset \mathrm{int}(A + B)$. ∎

Remark 2.1. The converses of Propositions 2.1 and 2.2 are not always true. We give the following simple examples:

(i) Let $X := R^2$, $A := \{x = (x_1, x_2) \in R^2 : 0 \leq x_i \leq 1 \text{ for } i = 1, 2\} \cup \{x = (0, x_2) \in R^2 : -1 \leq x_2 \leq 0\}$, and $B := \{x = (x_1, 0) \in R^2 : 2 \leq x_1 \leq 3\}$. Then the converses of the conditions (2.1) and (2.2) are not true though A is connected with $\mathrm{int}A = \mathrm{cor}A \neq \emptyset$ and B is convex.

(ii) Let $X := R^2$, $A := \{x = (x_1, x_2) \in R^2 : 0 \leq x_i \leq 1 \text{ for } i = 1, 2\}$, and $B := \{x = (x_1, 0) \in R^2 : 0 \leq x_1 \leq 1\} \cup \{x = (1, x_2) \in R^2 : -1 \leq x_i \leq 0\} \cup \{x = (x_1, -1) \in R^2 : 0 \leq x_i \leq 1\}$. Then the converses of the conditions (2.1) and (2.2) are not true though A is convex with $\mathrm{int}A = \mathrm{cor}A \neq \emptyset$ and B is connected (but not convex).

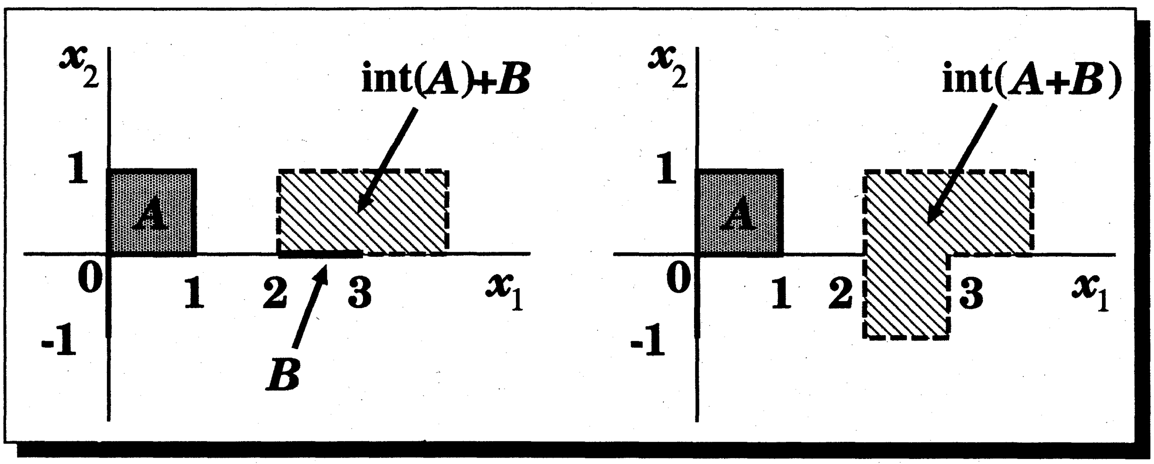

Figure 2.1: **The example of (i) of Remark 2.1.**

Next, we cite the following results without proofs.

Lemma 2.1. (See pages 10 and 59 in [2].) *The following statements hold:*

(i) *If A is a convex set in a linear space X and $p \in$* $\mathrm{cor}A$, *then* $[p, a) \subset \mathrm{cor}A$ *for any $a \in A$, $a \neq p$;*

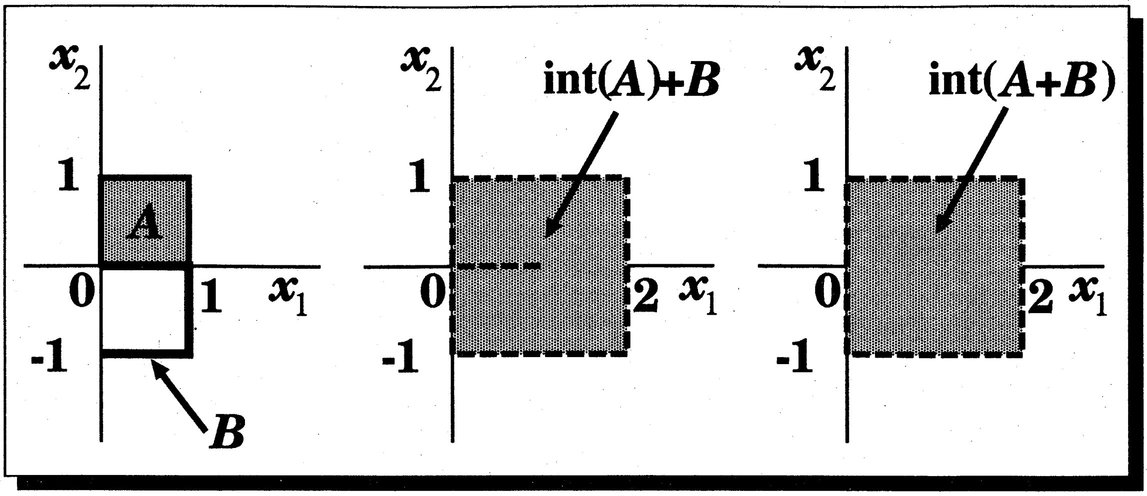

Figure 2.2: **The example of (ii) of Remark 2.1.**

(ii) *If A is a convex set in a linear topological space X and* $\text{int}A \neq \emptyset$, *then* $t\text{cl}A + (1-t)\text{int}A \subset \text{int}A$ *for* $0 \leq t < 1$.

By (ii) of Lemma 2.1 above, we have the following lemma.

Lemma 2.2. (See page 59 in [2].) *Let A be a set in a linear topological space X. If* $\text{int}A \neq \emptyset$ *then the following statements hold:*

(i) $\text{int}A \subset \text{cor}A$;

(ii) $\text{int}A = \text{cor}A$ *whenever A is convex.*

Remark 2.2. Without the convexity of A, (ii) of Lemma 2.2 is not always true. We give the following simple example: Let $X := R^2$, and $A := \{x = (x_1, x_2) \in R^2 : (x_1+1)^2 + x_2^2 \leq 1\} \cup \{x = (x_1, x_2) \in R^2 : (x_1-1)^2 + x_2^2 \leq 1\} \cup \{x = (0, x_2) \in R^2 : -\infty < x_2 < \infty\}$. Then, $(0,0) \in \text{cor}A$, $(0,0) \notin \text{int}A$, and hence $\text{int}A \neq \text{cor}A$.

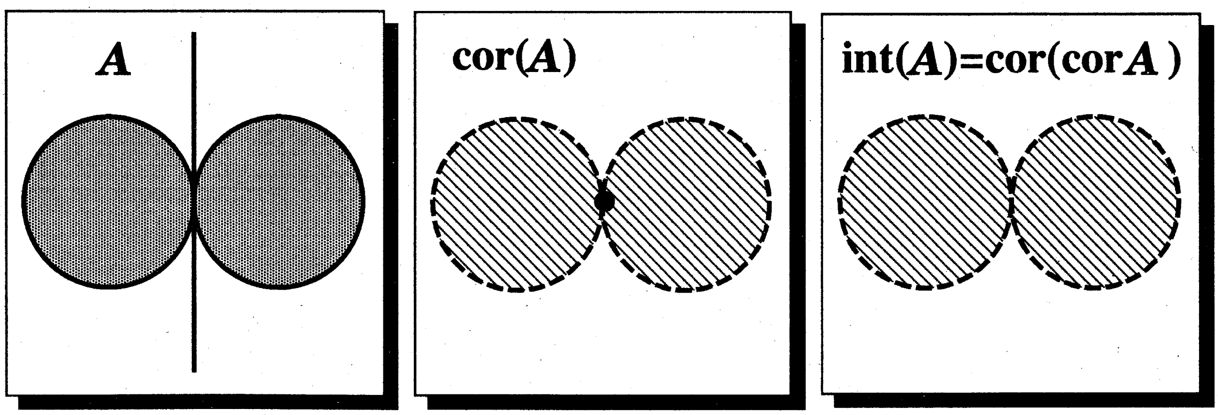

Figure 2.3: **An example with** $\text{int}A \neq \text{cor}A$ **and** $\text{cor}(\text{cor}A) \neq \text{cor}A$.

Now, we prove the main theorems.

Theorem 2.1. *Let A and B be nonempty convex sets in a linear space X. If* $\mathrm{cor}A \neq \emptyset$ *then*

$$\mathrm{cor}A + B \supset \mathrm{cor}(A+B), \tag{2.3}$$

and hence

$$\mathrm{cor}A + B = \mathrm{cor}(A+B). \tag{2.4}$$

Moreover, if $\mathrm{cor}B \neq \emptyset$ *then*

$$\mathrm{cor}A + \mathrm{cor}B = \mathrm{cor}A + B = A + \mathrm{cor}B = \mathrm{cor}(A+B). \tag{2.5}$$

Proof. For any $x \in \mathrm{cor}(A+B)$, there are $a \in A$ and $b \in B$ such that $x = a + b$. If $a \in \mathrm{cor}A$, then (2.3) holds. Let $a \notin \mathrm{cor}A$. Since $\mathrm{cor}A \neq \emptyset$, there exists a vector $p \in \mathrm{cor}A$, and so $[p, a) \subset \mathrm{cor}A$ by (i) of Lemma 2.1. For $2a - p + b \in X$, $x \in \mathrm{cor}(A+B)$ implies that there exists $0 < \lambda < 1$ such that $x_1 := \lambda(2a - p + b) + (1-\lambda)x \in A + B$. Let $x_2 := \lambda(p+b) + (1-\lambda)x$ then we have $\frac{x_2+x_1}{2} = x$ and $x_2 \in [p+b, x) = [p, a) + b$. Hence, $x_2 - b \in [p, a) \subset \mathrm{cor}A$. Since there are $\hat{a} \in A$ and $\hat{b} \in B$ such that $x_1 = \hat{a} + \hat{b}$, we have

$$x = \frac{x_2 + x_1}{2} = \frac{x_2 - b + (\hat{a} + \hat{b}) + b}{2} = \frac{x_2 - b + \hat{a}}{2} + \frac{\hat{b} + b}{2} \in \mathrm{cor}A + B$$

by (i) of Lemma 2.1 again. Thus, (2.3) is proved. So we have (2.4) by Proposition 2.1. Moreover, if $\mathrm{cor}B \neq \emptyset$ then using analogy of (2.4), we have $\mathrm{cor}A + \mathrm{cor}B = \mathrm{cor}(A + \mathrm{cor}B) = \mathrm{cor}(\mathrm{cor}(A+B))$. The last set is equivalent to $\mathrm{cor}(A+B)$ since $\mathrm{cor}(A+B)$ is nonempty convex. ∎

Theorem 2.2. *Let A and B be nonempty convex sets in a linear topological space X. If* $\mathrm{int}A \neq \emptyset$ *then*

$$\mathrm{int}A + B \supset \mathrm{int}(A+B), \tag{2.6}$$

and hence

$$\mathrm{int}A + B = \mathrm{int}(A+B). \tag{2.7}$$

Moreover, if $\mathrm{int}B \neq \emptyset$ *then*

$$\mathrm{int}A + \mathrm{int}B = \mathrm{int}A + B = A + \mathrm{int}B = \mathrm{int}(A+B). \tag{2.8}$$

Proof. Since A is a convex set with $\mathrm{int}A \neq \emptyset$, it follows from (ii) of Lemma 2.2 that $\mathrm{int}A = \mathrm{cor}A$. Also, we have $\mathrm{int}(A+B) = \mathrm{cor}(A+B)$ by the convexity of $A+B$. Thus, the conclusions (2.6), (2.7), and (2.8) follow from Theorem 2.1. ∎

Corollary 2.1. *The following statements hold:*

(i) *Let $A_1, \ldots, A_n$ be nonempty convex sets in a linear space X, and $\alpha_1, \ldots, \alpha_n$ scalars, not all zero. If* $\mathrm{cor}A_i \neq \emptyset$ *$(i = 1, \ldots, n)$ then*

$$\mathrm{cor}\left(\sum_{i=1}^{n} \alpha_i A_i\right) = \sum_{i=1}^{n} \alpha_i \mathrm{cor}A_i. \tag{2.9}$$

(ii) *Let $A_1, \dots, A_n$ be nonempty convex sets in a linear topological space X, and $\alpha_1, \dots, \alpha_n$ scalars, not all zero. If* $\mathrm{int}A_i \neq \emptyset$ *($i = 1, \dots, n$) then*

$$\mathrm{int}\left(\sum_{i=1}^{n} \alpha_i A_i\right) = \sum_{i=1}^{n} \alpha_i \mathrm{int} A_i. \tag{2.10}$$

The conclusion of (2.9) [resp. (2.10)] shows that the core operator [resp. the interior operator] behaves like a linear mapping on the family $C(X)$ of nonempty convex subsets of a linear space [resp. a linear topological space] X.

Remark 2.3. The properties (2.7), (2.8), and (2.10) may be useful for simplifying expressions of sets in $C_0(X)$ or $C(X)$. For instance, using them we get

$$\mathrm{int}A + \mathrm{cl}B = \mathrm{int}(A + B) \tag{2.11}$$

for any $A \in C_0(X)$ and $B \in C(X)$. Because $\mathrm{int}A + \mathrm{cl}B = \mathrm{int}(\mathrm{cl}A) + \mathrm{cl}B = \mathrm{int}(\mathrm{cl}A + \mathrm{cl}B) \subset \mathrm{int}(\mathrm{cl}(A + B)) = \mathrm{int}(A + B) \subset \mathrm{int}(A + \mathrm{cl}B) = \mathrm{int}A + \mathrm{cl}B$ by (2.7), Remark 1.1, and the property

$$\mathrm{cl}A + \mathrm{cl}B \subset \mathrm{cl}(A + B),$$

see page 40 in [1].

Moreover, we can generalize the previous theorems to the followings.

Theorem 2.3. *Let A and B be sets in a linear space X. If A is convex with* $\mathrm{cor}A \neq \emptyset$ *and* $\mathrm{cor}A + B$ *is midconvex, then*

$$\mathrm{cor}A + B \supset \mathrm{cor}(A + B), \tag{2.12}$$

and hence

$$\mathrm{cor}A + B = \mathrm{cor}(A + B). \tag{2.13}$$

Proof. For any $x \in \mathrm{cor}(A + B)$, there are $a \in A$ and $b \in B$ such that $x = a + b$. If $a \in \mathrm{cor}A$, then (2.12) holds. Let $a \notin \mathrm{cor}A$. Since $\mathrm{cor}A \neq \emptyset$, there exists a vector $p \in \mathrm{cor}A$, and so $[p, a) \subset \mathrm{cor}A$ by (i) of Lemma 2.1. For $2a - p + b \in X$, $x \in \mathrm{cor}(A + B)$ implies that there exists $0 < \lambda < 1$ such that $x_1 := \lambda(2a - p + b) + (1 - \lambda)x \in A + B$. Let $x_2 := \lambda(p + b) + (1 - \lambda)x$ then we have $x_2 \in [p + b, x) = [p, a) + b$, and hence $x_2 - b \in [p, a) \subset \mathrm{cor}A$. Since there are $\hat{a} \in A$ and $\hat{b} \in B$ such that $x_1 = \hat{a} + \hat{b}$, we have

$$x = \frac{x_2 + x_1}{2} = \frac{x_2 - b + (\hat{a} + \hat{b}) + b}{2},$$

and

$$\frac{x_2 - b + \hat{a}}{2} + \hat{b}, \frac{x_2 - b + \hat{a}}{2} + b \in \mathrm{cor}A + B$$

by (i) of Lemma 2.1 again. Since $\mathrm{cor}A + B$ is midconvex, we obtain

$$x = \frac{1}{2}\left\{\left(\frac{x_2 - b + \hat{a}}{2} + \hat{b}\right) + \left(\frac{x_2 - b + \hat{a}}{2} + b\right)\right\} \in \mathrm{cor}A + B. \tag{2.14}$$

Thus, (2.12) is proved. So we have (2.13) by Lemma 2.1. ∎

The essential point of the proof of Theorem 2.3 is (2.14).

Theorem 2.4. *Let A and B be sets in a linear topological space X. If A is convex with* $\mathrm{int}A \neq \emptyset$ *and* $\mathrm{int}A + B$ *is midconvex, then*

$$\mathrm{int}A + B \supset \mathrm{int}(A + B), \tag{2.15}$$

and hence

$$\mathrm{int}A + B = \mathrm{int}(A + B). \tag{2.16}$$

Proof. Since A is a convex set with $\mathrm{int}A \neq \emptyset$, it follows from (ii) of Lemma 2.2 that $\mathrm{int}A = \mathrm{cor}A$. Hence, we have

$$\mathrm{int}(A + B) \subset \mathrm{cor}(A + B) \subset \mathrm{cor}A + B = \mathrm{int}A + B,$$

by (i) of Lemma 2.2 and Theorem 2.3. Thus, (2.15) is proved, and hence (2.16) follows from Lemma 2.2. ∎

Theorems 2.3 and 2.4 are generalizations of Theorems 2.1 and 2.2, respectively. In fact, from the proposition below, it follows that Theorems 2.3 and 2.4 coincide with (i) and (ii) of Corollary 2.2 in the following, respectively.

Proposition 2.3. *Let A and B be nonempty sets in a linear space X. If A is convex with* $\mathrm{cor}A \neq \emptyset$*, then the following assertions are equivalent to each other:*

(i) $\mathrm{cor}A$ *is B-convex, i.e.,* $\mathrm{cor}A + B$ *is convex;*

(ii) $\mathrm{cor}A + B$ *is midconvex.*

Proof. Clearly (i) implies (ii). Conversely, let x and y be distinct two points of $\mathrm{cor}A+B$, then there are $x_a, y_a \in \mathrm{cor}A$ and $x_b, y_b \in B$ such that $x = x_a + x_b$ and $y = y_a + y_b$. Since $x_a, y_a \in \mathrm{cor}A$, by (i) of Lemma 2.1, there is $0 < \delta < 1$ such that line segment subsets $L := [x_a, x_a + \delta(y - x))$ and $R := (y_a + \delta(x - y), y_a]$ are contained in $\mathrm{cor}A$. Hence, $L_0 := L + x_b$ and $R_0 := R + y_b$ are included in $\mathrm{cor}A + B$. Using the midconvexity of $\mathrm{cor}A + B$, we have

$$I_0^{(1)} := \frac{1}{2}L_0 + \frac{1}{2}R_0 = \left(\frac{1+\delta}{2}x + \frac{1-\delta}{2}y, \frac{1-\delta}{2}x + \frac{1+\delta}{2}y\right) \subset \mathrm{cor}A + B.$$

These line segments L_0, R_0, $I_0^{(1)}$ are parts of the line segment $[x, y]$, and can be obtained by translating the line segment $\delta(x - y)$, in other words, each of them is an interval in the line segment $[x, y]$ with the ratio $\delta : 1$. Similarly, a double sequence of intervals $I_n^{(m)} \subset \mathrm{cor}A+B$ in the line segment $[x, y]$ with the ratio $\delta : 1$ can be defined recursively. Then for n large enough so that $(2^{n+1} + 1)\delta > 1$, we have

$$\left(\bigcup_{i=0}^{n} \bigcup_{j=1}^{2^i} I_i^{(j)}\right) \cup L_0 \cup R_0 = [x, y].$$

This shows that $\mathrm{cor}A + B$ is a convex set. ∎

Corollary 2.2. *The following statements hold:*

(i) *Let A and B be sets in a linear space X. If A is convex with* $\mathrm{cor}A \neq \emptyset$ *and* $\mathrm{cor}A$ *is B-convex, then* (2.12) *and* (2.13) *hold.*

(ii) *Let A and B be sets in a linear topological space X. If A is convex with* $\mathrm{int}A \neq \emptyset$ *and* $\mathrm{int}A$ *is B-convex, then* (2.15) *and* (2.16) *hold.*

Secondly, we give another condition assuring $\mathrm{cor}A + B = \mathrm{cor}(A + B)$.

Theorem 2.5. *Let A and B be sets in a linear space X. If B is algebraic open, i.e., $B = \mathrm{cor}B$, then*

$$A + \mathrm{cor}B = \mathrm{cor}(A + B) = A + B. \tag{2.17}$$

Moreover, if A is convex with $\mathrm{cor}A \neq \emptyset$, *then*

$$A + B = \mathrm{cor}A + B = \mathrm{cor}A + \mathrm{cor}B. \tag{2.18}$$

Proof. By Proposition 2.1 and $B = \mathrm{cor}B$, we have

$$A + \mathrm{cor}B \subset \mathrm{cor}(A + B) \subset A + B = A + \mathrm{cor}B,$$

which implies (2.17). Next, let A be convex with $\mathrm{cor}A \neq \emptyset$, and we show that $A + B = \mathrm{cor}A + B$. For any $x \in A + B$, there are $a \in A$ and $b \in B$ such that $x = a + b$. If $a \in \mathrm{cor}A$, then $x \in \mathrm{cor}A + B$ holds. Let $a \notin \mathrm{cor}A$. Since $\mathrm{cor}A \neq \emptyset$, there exists a vector $p \in \mathrm{cor}A$, and so $[p, a) \subset \mathrm{cor}A$ by (i) of Lemma 2.1. For $a + b - p \in X$, $b \in B = \mathrm{cor}B$ implies that there exists $0 < \lambda < 1$ such that $x_1 := \lambda(a + b - p) + (1 - \lambda)b = b + \lambda(a - p) \in B$. Let $x_2 := a + \lambda(p - a)$, then we have $x_1 + x_2 = a + b = x$, and also $x_1 \in B$ and $x_2 \in \mathrm{cor}A$. This shows that $x \in \mathrm{cor}A + B$. Then, $A + B \subset \mathrm{cor}A + B$, and hence we have $A + B = \mathrm{cor}A + B$. Thus, (2.18) is proved. ∎

Similarly, the following theorem can be proved.

Theorem 2.6. *Let A and B be sets in a linear topological space X. If B is topological open, i.e., $B = \mathrm{int}B$, then*

$$A + \mathrm{int}B = \mathrm{int}(A + B) = A + B. \tag{2.19}$$

Moreover, if A is convex with $\mathrm{int}A \neq \emptyset$, *then*

$$A + B = \mathrm{int}A + B = \mathrm{int}A + \mathrm{int}B. \tag{2.20}$$

Let $O_1(X)$ [resp. $O_2(X)$] be the family of nonempty algebraic open [resp. topological open] subsets of a linear space [resp. a linear topological space] X. Also, let $C_1(X)$ [resp. $C_2(X)$] be the family of convex subsets with nonempty core [resp. nonempty interior] of a linear space [resp. a linear topological space] X. Then, by the theorems above and Corollary 2.1, we obtain the following results.

Corollary 2.3. *The following statements hold:*

(i) *Let X be a linear space, and let $A_1, \ldots, A_n \in O_1(X) \cup C_1(X)$ and $\alpha_1, \ldots, \alpha_n$ scalars, not all zero. Then*

$$\mathrm{cor}\left(\sum_{i=1}^{n} \alpha_i A_i\right) = \sum_{i=1}^{n} \alpha_i \mathrm{cor}A_i. \tag{2.21}$$

(ii) *Let X be a linear topological space, and let $A_1, \ldots, A_n \in O_2(X) \cup C_2(X)$ and $\alpha_1, \ldots, \alpha_n$ scalars, not all zero. Then*

$$\mathrm{int}\left(\sum_{i=1}^{n} \alpha_i A_i\right) = \sum_{i=1}^{n} \alpha_i \mathrm{int}A_i. \tag{2.22}$$

3. OPERATORS BEHAVING LIKE LINEAR MAPPINGS

The conclusion of (2.21) [resp. (2.22)] shows that the core operator [resp. the interior operator] behaves like a linear mapping on the family $O_1(X) \cup C_1(X)$ [resp. $O_2(X) \cup C_2(X)$].

In order to increase understanding, let us investigate some families on which the closure and convex-hull operators behave like linear mappings. As shown in page 6 in [2], it is well-known that

$$\mathrm{co}(\alpha A + \beta B) = \alpha \mathrm{co} A + \beta \mathrm{co} B, \tag{3.1}$$

where A, B are nonempty subsets of a linear space X and α, β are scalars. Let $F(X)$ be the family of nonempty relatively compact subsets of a linear topological space X. If X is Hausdorff, then $\alpha \mathrm{cl}(A) + \beta \mathrm{cl}(B)$ is closed for any $A, B \in F(X)$ and scalars α, β, and hence

$$\mathrm{cl}(\alpha A + \beta B) = \alpha \mathrm{cl} A + \beta \mathrm{cl} B. \tag{3.2}$$

Also, given a subset A of a linear topological space X, the asymptonic cone of A, written $\mathrm{As}(A)$, is the set of all points $a = \lim t_\lambda x_\lambda$ for some nets $x_\lambda \in A$ and $t_\lambda > 0$ converging to 0, which implies that $\mathrm{As}(A) = \bigcap_{t>0} \mathrm{cl}\{\alpha x : x \in A, 0 < \alpha \leq t\}$; see [3] and [4]. Then, when A is a cone, we can easily verify that $\mathrm{As}(A) = \mathrm{cl} A$. Hence, in the same way as the proof of Theorem 2.12 in [3], we have the following: if A and B are nonempty cones with $\mathrm{As}(A) \cap -\mathrm{As}(B) = \{0\}$, and if one of them is convex and locally compact, then

$$\mathrm{cl}(A + B) = \mathrm{cl} A + \mathrm{cl} B. \tag{3.3}$$

Given a pointed (i.e., $C \cap (-C) = \{0\}$) convex cone C, let $K_C(X)$ be the family of nonempty convex, locally compact cones, included in C, in a linear topological space X. Then, from (3.3) it follows that (3.2) holds for any $A, B \in K_C(X)$ and scalars α, β, and hence it is true for any $A, B \in F(X) \cup K_C(X)$ and scalars α, β whenever X is Hausdorff, since the sum of a compact set and a closed set is closed in X. Therefore, for any pointed closed cone C, we can see that three operators int, cl, co behave like linear mappings on the family $(O_2(X) \cup C_2(X)) \cap (F(X) \cup K_C(X))$ in a Hausdorff linear topological space X.

Moreover, as presented in Section 2, we have $\mathrm{int}A + \mathrm{int}A = \mathrm{int}A + A = A + \mathrm{int}A = 2\mathrm{int}A$, $\mathrm{int}A + \mathrm{cl}A = \mathrm{cl}A + \mathrm{int}A = 2\mathrm{int}A$, and $\mathrm{cl}A + \mathrm{cl}A = 2\mathrm{cl}A$ for any $A \in C_2(X)$. Also, as shown in page 59 in [2], we note that $\mathrm{int}(\mathrm{int}A) = \mathrm{int}A$, $\mathrm{cl}(\mathrm{cl}A) = \mathrm{cl}A$, $\mathrm{int}(\mathrm{cl}A) = \mathrm{int}A$, and $\mathrm{cl}(\mathrm{int}A) = \mathrm{cl}A$ for any $A \in C_2(X)$. These can be interpreted in the following way. Let $\mathrm{int}(A) := \mathrm{int}A$, $\mathrm{cl}(A) := \mathrm{cl}A$, $\mathrm{co}(A) := \mathrm{co}A$ for $A \subset X$, and let $(\mathrm{int} + \mathrm{cl})(A) := \mathrm{int}(A) + \mathrm{cl}(A)$, $(\alpha\mathrm{int})(A) := \alpha(\mathrm{int}A)$, $(\mathrm{int} \circ \mathrm{cl})(A) := \mathrm{int}(\mathrm{cl}(A))$, etc. Then we can give the following relation tables on the family of closed convex subsets with nonempty interior, where the binary relations $+$ and $\circ$ mean (row)+(column) and (row)$\circ$(column), respectively. Whenever limited to the convex cones, the scalar multipliers 2 on the left-hand table are superfluous.

$+$	int	cl	co
int	2int	2int	2int
cl	2int	2cl	2cl
co	2int	2cl	2co

$\circ$	int	cl	co
int	int	int	int
cl	cl	cl	cl
co	int	cl	co

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